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VECTORIAL LINEAR CONNECTIONS

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Abstract. In this article, we consider a vectorial linear connection which is determined by three fixed vector fields. Classifying these vectorial connections, we obtain a new type of generalized statistical manifolds which allow non-zero torsion.

1. Introduction

In an Euclidean space R^d, there is a canonical way to identify the tangent spaces at different points, namely giving a parallel displacement of a tangent plane. Using this parallel displacement, we can define the derivative of a vector field in a given direction.

On the other hand, in a Riemannian manifold (M, g), there is no canonical way of identifying tangent spaces. Thus we need a linear connection ∇, a notion of derivative for vector fields depending on certain choices.

A linear connection ∇ has its dual connection ∇^* with respect to the metric g, satisfying

$$d(g(X, Y)) = g(\nabla X, Y) + g(X, \nabla^* Y),$$

for all vector fields X, Y, Z. This notion of dual connections is introduced by A.P. Norden, Nagaoka and Amari, for details we refer to ([2, 10, 11]).

A metric connection is a linear connection ∇ which satisfies $\nabla = \nabla^*$, that is $\nabla g = 0$.

A statistical manifold is a manifold (M, g, ∇) which satisfies $T^\nabla = T^{\nabla^*} = 0$, where T^∇ denotes the torsion of the connection ∇. A notion

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for generalized statistical manifolds, which allow a non-zero torsion, is introduced in [8].

Classifying the metric connections, we can express a vectorial metric connection as follows ([1, 5]):

$$\nabla_X Y = \nabla^g_X Y + g(X,Y)V - g(V,Y)X \tag{1.1}$$

for the Levi-Civita connection ∇^g and a fixed vector field V.

Generalizing the above expression by two fixed vector fields, we can define a vectorial linear connection and obtain examples for generalized statistical manifolds, for details please refer to [7] and section 2 of this article.

In this article, we generalize the expression (1.1) by three fixed vector fields and obtain new examples for generalized statistical manifolds as defined in [8] (section 3).

2. Preliminaries

Let (M,g) be a Riemannian manifold and $\Gamma(M)$ denote the set of sections of the tangent bundle TM.

A metric connection ∇ is a linear connection satisfying

$$V(g(X,Y)) = g(\nabla_V X, Y) + g(X, \nabla_V Y) \tag{2.1}$$

for $V, X, Y \in \Gamma(M)$.

The Levi-Civita connection, denoted by ∇^g, is the unique metric connection with torsion $T^\nabla = 0$, where the torsion tensor T^∇ of a linear connection ∇ is defined by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

where $[X,Y]$ is the Lie-bracket.

The difference between a linear connection ∇ and the Levi-Civita connection ∇^g is a $(2,1)$-tensor field A satisfying that

$$\nabla_X Y = \nabla^g_X Y + A(X,Y).$$

The same notation will be used for the $(3,0)$- tensor field derived from the $(2,1)$-tensor A as follows:

$$A(X,Y,Z) = \langle A(X,Y), Z\rangle.$$

The properties of A, being symmetric or antisymmetric, give geometric interpretations for the connection ∇. We can easily check the following.

REMARK 2.1. A connection $\nabla = \nabla^g + A$ is

(i) metric if and only if $A(X, Y, Z) + A(X, Z, Y) = 0$,
(ii) torsion-free if and only if $A(X, Y, Z) - A(Y, X, Z) = 0$.

In particular, if $A(X, Y, Z)$, as a $(3, 0)$-tensor, is totally symmetric with respect to X, Y, Z, then (M, g, ∇) is a statistical manifold, whose torsion is necessarily zero, for details we refer to [4].

Using the $O(n)$ action on A, we classify metric connections, actually the space of tensor fields A's as follows ([1, 5, 12])

$$\mathcal{A} = TM \oplus \Lambda^3(TM) \oplus \mathcal{A}'. \tag{2.2}$$

The first space of the above decomposition gives the so-called vectorial metric connections which can be expressed as

$$A(X, Y) = g(X, Y)V - g(V, Y)X,$$

for a fixed vector field V.

In [7], taking two fixed vector fields V_1, V_2, another type of vectorial connections is discussed, where the tensor A is defined by

$$A(X, Y) = g(X, Y)V_1 - g(V_2, Y)X.$$

Given a linear connection ∇, a linear connection ∇^* can be derived uniquely such that the metric g is preserved by the connections ∇ and ∇^* with respect to the metric g, that is

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y).$$

Here the connection ∇^* is expressed as

$$\nabla^*_Z X = \nabla^g_Z X + A^*(Z, X)$$

where

$$A(Z, X, Y) + A^*(Z, Y, X) = 0.$$

3. Vectorial linear connections

Now we consider a $(3, 0)$- tensor field $\tilde{A}(X, Y, Z)$ determined by three fixed vector fields V_1, V_2, V_3 as follows:

$$\tilde{A}(X, Y, Z) = g(X, V_1)g(Y, Z) + g(Y, V_2)g(X, Z) + g(Z, V_3)g(X, Y).$$

Then a vectorial linear connection $\tilde{\nabla}$ by three fixed vector fields can be defined by

$$\tilde{\nabla}_X Y = \nabla^g_X Y + \tilde{A}(X, Y) \tag{3.1}$$

with

$$\tilde{A}(X,Y) = g(X,V_1)Y + g(Y,V_2)X + g(X,Y)V_3.$$

We can derive following geometric properties of $\tilde{\nabla}$.

PROPOSITION 3.1. *(i) A vectorial connection $\tilde{\nabla}$ by three vector fields V_1, V_2, V_3 is a metric connection if and only if $V_1 = 0$ and $V_2 = -V_3$.*

(ii) A vectorial connection $\tilde{\nabla}$ by three vector fields V_1, V_2, V_3 is torsion-free if and only if $V_1 = V_2$.

Proof. By Remark 2.1, we compute the following.

(i) For all $X, Y, Z \in \Gamma(TM)$

$$\begin{aligned}\tilde{A}(X,Y,Z) + \tilde{A}(X,Z,Y) &= g(X,V_1)g(Y,Z) + g(Y,V_2)g(X,Z) + g(Z,V_3)g(X,Y)\\ &\quad + g(X,V_1)g(Z,Y) + g(Z,V_2)g(X,Y) + g(Y,V_3)g(X,Z)\\ = \ & 2g(X,V_1)g(Y,Z) + g(Y,V_2+V_3)g(X,Z) + g(X,Y)g(Z,V_2+V_3) = 0,\end{aligned}$$

if and only if $V_1 = 0$ and $V_2 + V_3 = 0$.

(ii) For all $X, Y, Z \in \Gamma(TM)$

$$\begin{aligned}\tilde{A}(X,Y,Z) - \tilde{A}(Y,X,Z) &= g(X,V_1)g(Y,Z) + g(Y,V_2)g(X,Z) + g(Z,V_3)g(X,Y)\\ &\quad - g(Y,V_1)g(X,Z) - g(X,V_2)g(Y,Z) - g(Z,V_3)g(Y,X)\\ = \ & g(X,V_1-V_2)g(Y,Z) + g(Y,V_2-V_1)g(X,Z) = 0,\end{aligned}$$

if and only if $V_1 = V_2$.

□

PROPOSITION 3.2. *The dual connection of a vectorial connection $\tilde{\nabla}$ by three vector fields V_1, V_2, V_3 is*

$$\tilde{\nabla}^* = \nabla^g + \tilde{A}^*$$

with

$$\tilde{A}^*(X,Y) = g(X,-V_1)Y + g(Y,-V_3)X + g(X,Y)(-V_2).$$

Proof. For a vectorial connection $\tilde{\nabla} = \nabla^g + \tilde{A}(X,Y)$ by three vector fields V_1, V_2, V_3 we have

$$\tilde{A}(X,Y) = g(X,V_1)Y + g(Y,V_2)X + g(X,Y)V_3.$$

For the dual connection $\tilde{\nabla}^* = \nabla^g + \tilde{A}^*$ of $\tilde{\nabla}$, we can compute

$$\begin{aligned}&\tilde{A}^*(X,Y,Z) = -\tilde{A}(X,Z,Y)\\ &= \ -g(X,V_1)g(Z,Y) - g(Z,V_2)g(X,Y) - g(Y,V_3)g(X,Z)\\ &= \ g(g(X,-V_1)Y + g(Y,-V_3)X + g(X,Y)(-V_2), Z)\end{aligned}$$

so

$$\tilde{A}^*(X,Y) = g(X,-V_1)Y + g(Y,-V_3)X + g(X,Y)(-V_2).$$

□

A statistical manifold is a torsion-free manifold with some property. In [8], a notion for generalized statistical manifolds is introduced, where the torsion can be non-zero.

Definition 3.3 ([3, 8, 9]). A Riemannian manifold (M, g, ∇) is a statistical manifold admitting torsion if

$$(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = -g(T^\nabla(X,Y), Z) \tag{3.2}$$

for $X, Y, Z \in \Gamma(TM)$, where T^∇ is the torsion tensor of ∇.

In [7] an equivalent condition for (3.2) is introduced.

Proposition 3.4 ([7]). *A Riemannian manifold* (M, g, ∇) *where* $\nabla = \nabla^g + A$ *is a statistical manifold admitting torsion if and only if*

$$A(X,Y,Z) = A(Z,Y,X) \quad \text{for} \quad X,Y,Z \in \Gamma(TM).$$

In Weyl geometry we consider a class of conformal metrics and a torsion-free connection called Weyl connection which preserves the conformal structure. In particular, this connection is known to be uniquely constructed by a fixed vector field V. So, choosing a metric g in the conformal class of metrics we can define the Weyl connection as follows([6]).

Definition 3.5 (Weyl connection, [1, 6]). Given a Riemannian manifold (M, g) and a fixed vector field V, a Weyl connection ∇^w is then defined by

$$\nabla^w = \nabla^g + A^w$$

with

$$A^w(X,Y) = g(X,V)Y + g(Y,V)X - g(X,Y)V.$$

Now considering the vectorial connections as defined in (3.1) we will obtain some types of examples for Statistical manifolds admitting torsion in the next Theorem 3.6.

Theorem 3.6. *(i) Consider a vectorial connection* $\tilde{\nabla} = \nabla^g + \tilde{A}$ *by fixed vector fields* V_1, V_2, V_3*. Then a Riemannian manifold* $(M, g, \tilde{\nabla})$ *is a statistical manifold admitting torsion if and only if*

$$V_1 = V_3.$$

(ii) The dual connection $(\nabla^w)^*$ *of a Weyl connection* ∇^w *is a statistical manifold admitting torsion.*

Proof. (i) From Proposition 3.4 we obtain

$$
\begin{aligned}
A(X,Y,Z) - A(Z,Y,X) &= g(X,V_1)g(Y,Z) + g(Y,V_2)g(X,Z) + g(Z,V_3)g(X,Y) \\
&\quad -g(Z,V_1)g(Y,X) - g(Y,V_2)g(Z,X) - g(X,V_3)g(Z,Y) \\
&= g(X,V_1 - V_3)g(Y,Z) + g(Z,V_3 - V_1)g(X,Y) \\
&= 0.
\end{aligned}
$$

So, this is zero for all X, Y, Z if and only if $V_1 = V_3$.

(ii) Consider a Weyl connection $\nabla^w = \nabla^g + A^w$ where for a fixed vector field V

$$A^w(X,Y) = g(X,V)Y + g(Y,V)X - g(X,Y)V.$$

By Proposition 3.2, it holds $(\nabla^w)^* = \nabla^g + (A^w)^*$ with

$$(A^w)^*(X,Y) = g(X,-V)Y + g(Y,V)X + g(X,Y)(-V).$$

From the above assertion *(i), the manifold* $(M, g, (\nabla^w)^*)$ *is a statistical manifold admitting torsion.*

□

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